cati equation can be used, which greatly reduces the computation burden. Also, not shown here, the linear differential equation in Eq. (7) "tracks" the actual state trajectories and serves to force the nominal-model state trajectories to the actual state trajectories.

Finally, if W is chosen too small from the optimal weighting matrix, the resulting matrix Riccati equations are "stiff" and tend to diverge very quickly. This can be extremely useful in determining a good starting guess for the weighting matrix.

Conclusions

This Note has established a matrix Riccati solution for the minimum model error estimation algorithm. The functional form for the solution of the associated two-point boundary value problem includes one Riccati equation and one linear differential equation with discrete update discontinuities at each measurement time. The homogeneous Riccati equation can be reduced to an algebraic Riccati equation if the assumed model is linear and time-invariant, thus reducing the solution to linear differential and algebraic equations. The algorithm was demonstrated for a simple linear time-invariant model. Results indicate that the algorithm is capable of determining optimal state estimates by using the closed-form solution.

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Optimal Control System Design with Prescribed Damping and Stability Characteristics

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Introduction

DESIGN of multivariable optimal control systems with specified damping and stability characteristics has been a goal pursued by a number of researchers. Practical time-domain performance requirements often include the following two important specifications: 1) the response must be sufficiently fast and smooth, and 2) the response must not exhibit excessive overshoot and oscillations. The first specification places a bound on the settling time, whereas the second one gives rise to a bound on the damping ratio. For this reason, the shaded area of Fig. 1 has been widely accepted as a suitable design sector. Because direct optimal pole placement in the shaded area is a very difficult problem to solve, a multitude of approximate regions have been proposed in the literature.[1-5] The following theorem has been crucial in developing most of the root-clustering algorithms.

Relative Stability Theorem: The eigenvalues of the matrix A lie within the shaded stability region of Fig. 2, if and only if the eigenvalues of the $2N \times 2N$ matrix

$$A_\alpha \otimes \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \tag{1}$$

have negative real parts. The angle β is given by $\beta = \pi/2 - \theta$, and A_α is defined as $A - \alpha I$. See Ref. 6 for the proof.

For example, a design procedure based on the preceding theorem recently appeared in Ref. 3. The straightforward implementation of this theorem, however, results in a root-clustering sector that underestimates the design region as can be seen in Fig. 2. Hence, the main goal of this Note is to develop a new design method based on a more accurate approximation of the shaded region of Fig. 1.

Problem Formulation and Solution

Consider the following linear time-invariant dynamic system

$$\dot{X} = AX + BU \tag{2}$$

where A and B are constant matrices of order $N \times N$ and $N \times M$, respectively. The problem solved in this Note is to determine a state feedback controller of the form $U = KX$ such that 1) the closed-loop system matrix $A + BK$ has all of its eigenvalues within the shaded region of Fig. 1, and 2) the following linear quadratic performance index is minimum

$$J = \int_0^\infty (X^T Q X + 2X^T M U + U^T R U)\,\mathrm{d}t$$

The preceding eigenvalue-clustering problem is solved by transforming the shaded area into the left-hand plane of an associated dynamic system. To begin with the transformation process, consider the Laplace transform of Eq. (2), assuming zero initial conditions:

$$sX(s) = AX(s) + BU(s) \tag{3}$$

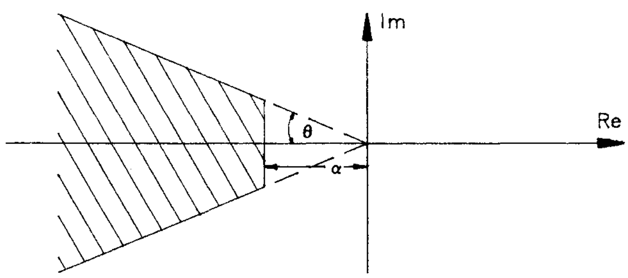

Fig. 1 Design sector with prescribed damping and stability specifications.

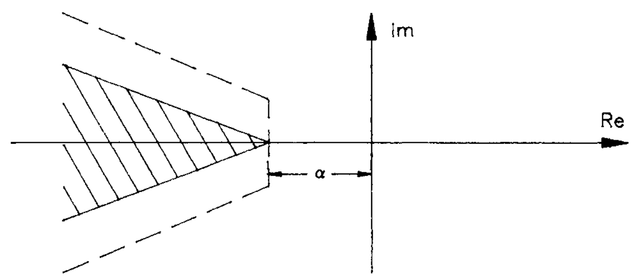

Fig. 2 Design sector resulting from direct implementation of the relative stability theorem.

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Let

$$s = ½(p + \alpha^2/p) \tag{4}$$

Then Eq. (3) becomes

$$pX(p) = 2AX(p) - \frac{\alpha^2}{p}X(p) + 2BU(p) \tag{5}$$

or in the time domain

$$\dot{X} = 2AX - \alpha^2\int X\,\mathrm{d}t + 2BU \tag{6}$$

Define two new state variables

$$w_1 = \int X\,\mathrm{d}t, \qquad w_2 = X \tag{7}$$

Hence, the dynamics of the associated system are governed by

$$\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\end{bmatrix} = \begin{bmatrix}0 & I\\ -\alpha^2 I & 2A\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix} + \begin{bmatrix}0\\ 2B\end{bmatrix}U \tag{8}$$

or

$$\dot{W} = HW + FU \tag{9}$$

Now the relationship between the original and associated dynamic system eigenvalues can be established. Set $s = x + yj$ and $p = \mathrm{R}e^{j\theta}$. Then Eq. (4) becomes

$$x + yj = ½(\mathrm{R}e^{j\theta} + \alpha^2/\mathrm{R}e^{j\theta}) \tag{10}$$

Equating the real and imaginary parts results in

$$\begin{aligned} x &= ½(r + \alpha^2/r)\cos\theta \\ y &= ½(r - \alpha^2/r)\sin\theta \end{aligned} \tag{11}$$

Further algebraic manipulation yields

$$\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = \alpha^2 \tag{12}$$

which is the equation of a hyperbola centered at the origin with foci on the real axis. The hyperbola is asymptotic to the lines

$$y = \mp x\tan\theta \tag{13}$$

Hence, the straight lines emanating from the origin at angles θ in the p plane are mapped into hyperbolas in the s plane.

Based on the preceding analytical results, it is possible to develop a procedure to design closed-loop systems with eigenvalues clustered within the shaded region of Fig. 1. First, however, note that the proposed transformation from the s domain to the p domain preserves the controllability of the system. This can be easily seen from the fact that the associated p-domain system is actually the s-domain system controlled by the state-space version of the proportional + integral (PI) controller as given next:

$$U = 2BKX - \alpha^2\int X\,\mathrm{d}t \tag{14}$$

This interpretation of the associated system gives a physical meaning to the s to p domain transformation. Clearly, if the s-domain system is controllable, i.e., $[A, B]$ is a controllable pair, then the associated p-domain system is also controllable, i.e., $[H, F]$ is a controllable pair.

Design Algorithm: The results of the preceding section can now be summarized in the form of a design algorithm.

1) Select a suitable pair $[\alpha, \theta]$ to represent the design region.
2) Form the associated p-domain closed-loop system matrix

$$H_{\mathrm{CL}} = \begin{bmatrix}0 & I\\ -\alpha^2 I & 2(A + BK)\end{bmatrix}$$

3) Rotate the H_{CL} matrix through $\beta = \pi/2 - \theta$ in the p plane using

$$E = H_{\mathrm{CL}} \otimes \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}$$

4) Because the integral gain is already fixed by a suitable choice of α, the problem, at this step, is reduced to the optimal output-feedback control of the associated p-domain system. Finding a proportional gain matrix K that stabilizes the E matrix and also minimizes the quadratic performance index

$$J = \int_0^\infty \left(W^T\begin{bmatrix}0 & 0\\ 0 & Q\end{bmatrix}W + 2W^T\begin{bmatrix}0\\ M\end{bmatrix}U + U^TRU\right)\mathrm{d}t$$

can be accomplished by employing a numerical algorithm in the spirit of Ref. 7. Please note that $X = w_2$. It now follows that if the $4N \times 4N$ rotation matrix E has all of its poles in the left-hand complex plane, then the poles of $A + BK$ lie in the design sector of Fig. 1.

Example: The dynamic system considered is a double integrator:

$$\dot{X} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}X + \begin{bmatrix}0\\ 1\end{bmatrix}U$$

The performance index matrices were chosen to be

$$Q = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad M = \begin{bmatrix}1.2\\ 1.2\end{bmatrix}, \qquad R = [1]$$

The standard linear quadratic regulator design locates the eigenvalues at $\lambda_{1,2} = -0.387 \pm 0.921j$ with the damping ratio of $\zeta = 0.387$. The sector defined by $\alpha = 1.0$ and $\theta = 60$ deg was selected as the design region. The algorithm developed in this paper produced a state feedback matrix with $k_{11} = -4.261$ and $k_{12} = -3.844$. The resulting closed-loop eigenvalues are $\lambda_{1,2} = -1.922 \pm 0.753j$ with $\zeta = 0.931$. In comparison, the procedure of Ref. 3 results in an overdamped closed-loop system with $k_{11} = -5.788$, $k_{12} = -5.163$, $\lambda_1 = -1.645$, and $\lambda_2 = -3.518$.

Conclusions

This Note has presented a new method to design optimal state-feedback controllers. The resulting closed-loop system has all of its eigenvalues clustered within a region defined by stability and damping requirements. The analytical results can be easily extended to include the dynamic output-feedback controller design problem. The major drawback of the present method is the fourfold increase in the computational memory requirements. However, availability of the supercomputer like personal workstations will quickly alleviate this problem in the near future. The focus of current research is on the use of the H_∞-based performance index to achieve robustness against parametric uncertainty.

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Design of Guaranteed Performance Controllers for Systems with Varying Parameters

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I. Introduction

SYSTEMS are designed to operate within a nominal domain that may cover different stages of a standard operation. Therefore, multiple models, or models with varying parameters, characteristic of the current operating conditions, must be established to represent the dynamics. However, the number of models and related control laws must be reduced to be tractable.

The problem of the design of guaranteed cost control laws has been a topic of interest since Chang and Peng[1] introduced the idea of modifying the Riccati equation of the standard linear quadratic regulator (LQR) problem to cope with parameter uncertainties. More recently, with the large emphasis in robust control theory, the topic has gained new interest with authors such as Vinkler and Wood,[2] Petersen and Hollot,[3] and Schmitendorf.[4]

In this Note, the results of Vinkler are extended to the case of a variable control matrix, and a new formulation of the modified Riccati equation is proposed. Guaranteed performance and stability domains are then derived around each reference point subject to such control laws. A paving of the whole operations domain is then possible using repetitive calculations. An heuristic approach is proposed to select a limited number of reference points. This approach is applied to the design of multiple laws for the longitudinal control of an airplane within its flight domain.

II. Guaranteed Cost Control Law

Let us consider the linear system given by

$$\dot{x} = A(p)x + B(p)u \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^m \tag{1}$$

with

$$A(p) = A_0 + \sum_{i=1}^{N} p_i A_i \qquad B(p) = B_0 + \sum_{i=1}^{N} p_i B_i \tag{2}$$

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where (A_0, B_0) is controllable and p is a vector of characteristic parameters of the system operating point, and matrices A_i are supposed rank one matrices. Let D_0 be the feasible operations domain. Then we consider $p \in D_p$ where D_p is a connex subset of D_0. The cost functional C over the entire operation domain is

$$C = \int_0^T (x^t Q x + u^t R u)\, \mathrm{d}t \tag{3}$$

where Q and R are, respectively, positive semidefinite and positive definite matrices. Let $S(t)$ be the $n \times n$ symmetric matrix solution of the modified Riccati equation defined as

$$\dot{S}(t) + S(t)A_0 + A_0^t S(t) - S(t)B_0 R^{-1} B_0^t S(t) + Q + P[S(t)] = 0 \tag{4}$$

with $0 \le t \le T$ and $S(T) = 0$ where matrix $P(S)$ is a symmetric upper bound of

$$E(p,S) = S[A(p) - A_0] + [A(p) - A_0]^t S + SB_0 R^{-1} B_0^t S - SB(p)R^{-1}B^t(p)S \tag{5}$$

in the sense that

$$x^t P(S) x \ge x^t E(p,S) x \qquad \forall p \in D_p \qquad \forall x \in \mathbb{R}^n \tag{6}$$

In Sec. III it will be shown how to find such an upper bound.

Here the following theorem holds:

Theorem: Let $S(t)$ be the solution of the modified Riccati equation (4). Then choosing the control law $u(t) = -R^{-1}B_0^t S(t)x(t)$, the value of the cost functional C is bounded above

$$\int_0^T (x^t Q x + u^t R u)\, \mathrm{d}t \le x_0^t S(0) x_0, \qquad \forall p \in D_p \tag{7}$$

So this control law is called a guaranteed cost control law over D_p.

Note that this theorem is a generalization of Theorem 1 in Ref. 2, because here we also consider uncertainty in the control matrix B.

Proof: From Eq. (4) we get

$$\forall x \in \mathbb{R}^n : x^t[\dot{S} + SA_0 + A_0^t S - SB_0 R^{-1} B_0^t S + Q + P(S)]x = 0 \tag{8}$$

and replacing $P(S)$ by $E(p,S)$, the following inequality is obtained:

$$\forall x \in \mathbb{R}^n : x^t[\dot{S} + SA(p) + A^t(p)S - SB(p)R^{-1}B^t(p)S + Q]x \le 0 \tag{9}$$

From Eq. (1) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^t S x) = x^t[\dot{S} + SA(p) + A^t(p)S]x + x^t[SB(p)]u + u^t[B^t(p)S]x \tag{10}$$

and from Eq. (8)

$$\forall x \in \mathbb{R}^n : x^t[\dot{S} + SA(p) + A^t S]x \le x^t[SB(p)R^{-1}B^t(p)S - Q]x \tag{11}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^t S x) \le x^t[SB(p)R^{-1}B^t(p)S - Q]x + x^t SB(p)u + u^t B^t S x \tag{12}$$